GCE: 502, Linear Algebra

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Exercise 1:

Let K be $\mathbb{R}$ or $\mathbb{C}$ and $a_0, ..., a_{n-1}$ be in K. Let C be the n by n matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix}$$

Show that the characteristic polynomial of C and the minimal polynomial of C are both equal to $P(t) = a_0 + a_1 t + \cdots + a_{n-1}t^{n-1} + t^n$.

Exercise 2:

Let A and B be two invertible n by n matrices. Let M be the matrix

$$M = \begin{pmatrix} A & B \\ B^{-1} & A^{-1} \end{pmatrix}.$$

Assume that M has rank n. Show that A and B commute.

Exercise 3:

Let A be an n by n invertible matrix with entries in K. Suppose that u and v are two vectors in K^n such that $1 + v^T A^{-1} u \neq 0$. Show that $A + uv^T$ is invertible.

Hint : It suffices to prove that the inverse of $A + uv^T$ is

$$A^{-1} - mA^{-1}uv^T A^{-1}$$

where m is an adequate scalar.

Exercise 4:

Let $A = (a_{ij})$ be a symmetric n by n matrix with real entries. Let $\lambda_1, ..., \lambda_n$ be the eigenvalues of A. Show that

$$\sum_{1 \leq i,j \leq n} a_{ij}^2 = \sum_{i=1}^{n} \lambda_i^2$$